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Introduction. The elliptic functions are considered as the inverse of the elliptic integrals originally based in the works of Abel and Jacobi. An elliptic function is a meromorphic function, allowing for the periods, all of which can be formed by addition and subtraction of the two initial periods of w_1 and w_2 having an imaginary relationship. In short, a meromorphic function is called elliptic, if it is doubly periodic with periods of w_1 and w_2 , for which there is an imaginary number. Such a function $f(u)$ satisfies the following[1]:

$$f(u+w_1) = f(u) \tag{1}$$

and

$$f(u+w_2) = f(u). \tag{2}$$

We shall always take w_1 and w_2 in clockwise order; that is we shall assume that $\frac{w_1}{w_2}$ has positive imaginary part. For a given lattice L, a meromorphic function on C is said to be an elliptic function relative to L, if $f(u+l) = f(u)$ for all $l \in L$. It suffices to check this property $l = w_1$ and $l = w_2$. In other words, an elliptic function is periodic with two periods w_1 and w_2. Such a function is determined by its values on the fundamental parallelogram P. This fundamental parallelogram P can be defined by:

$$P = \{mw_1 + nw_2 | m, n \in Z\}. \tag{3}$$

We first observe that no elliptic function (except a constant) can be bounded over the interior and perimeter of a period parallelogram, for if it were, it would be bounded over the whole complex plane and hence, by Liouville's theorem, would be constant. It must, accordingly, have some poles inside or on the parallelogram and the number of these must be finite for, otherwise, there would be a limit point of such singularities and this would be an essential singularity. It is always possible, therefore, to translate the parallelogram in the complex plane so as to create a cell for which there are no singularities on its perimeter. The number of poles within the cell (each multiple pole being counted according to its order) is termed the order of the elliptic function. The Jacobian functions each have a pair of simple poles within a cell and so are of second order. $P(u)$ has a double pole at each of the congruent points $u = 2mw_1 + 2nw_3$ and hence the Weierstrass function is also of order two. Its derivative $\wp'(u)$ has a triple pole at $u = 0$ and is of third order.

The main content of the work. An elliptic function is a meromorphic function that is periodic in two directions.The order of an elliptic function is never less than 2, so in terms of

[1]Lavrentev, M.A. Methods of the theory of functions of a complex variable / M. A. Lavrentev, B. V.Shabat. M. : "Nauka", 2002. P. 688-694.

singularities, the simplest elliptic functions are those of order 2. These can be divided into two classes: those which have a single irreducible double pole in each cell at which the residue is zero, and those which have two simple poles in each cell at which the two residues are equal in absolute value, but of opposite sign[2].

As a periodic function of a real variable is defined by its values on an interval, an elliptic function is determined by its values on a fundamental parallelogram, which then repeat in a lattice. In geometry and group theory, a lattice in R^n is a subgroup of R^n which is isomorphic to Z^n, and which spans the real vector space R^n . In other words, for any basis of R^n, the subgroup of all linear combinations with integer coefficients of the basis vectors forms a lattice. A lattice may be viewed as a regular tiling of a space by a primitive cell.

A fundamental pair of periods is an ordered pair of complex numbers that define a lattice in the complex plane. This type of lattice is the underlying object with which elliptic functions and modular forms are defined. The fundamental pair of periods is a pair of complex numbers $\omega_1, \omega_2 \in C$ such that their ratio $\frac{\omega_1}{\omega_2}$ is not real. In other words, considered as vectors in R^2, the two are not collinear.

The lattice generated by ω_1 and ω_2 is

$$P = \Lambda = \{m\omega_1 + n\omega_2 | m, n \in Z\}. \quad (4)$$

This lattice is also sometimes denoted as $\lambda(\omega_1, \omega_2)$ to make clear that it depends on ω_1 and ω_2. It is also sometimes denoted by Ω or $\Omega(\omega_1, \omega_2)$, or simply by (ω_1, ω_2). The two generators ω_1 and ω_2 are called the lattice basis. The parallelogram defined by the vertices 0, ω_1 and ω_2 is called the fundamental parallelogram.

A doubly periodic function cannot be holomorphic, as it would then be a bounded entire function, and by Liouville's theorem every such function must be constant. A holomorphic function is a complex-valued function of one or more complex variables that is complex differentiable in a neighborhood of every point in its domain. The existence of a complex derivative in a neighborhood is a very strong condition, for it implies that any holomorphic function is actually infinitely differentiable and equal to its own Taylor series (analytic). Holomorphic functions are the central objects of study in complex analysis. Though the term analytic function is often used interchangeably with "holomorphic function", the word "analytic" is defined in a broader sense to denote any function (real, complex, or of more general type) that can be written as a convergent power series in a neighborhood of each point in its domain. The fact that all holomorphic functions are complex analytic functions, and vice versa, is a major theorem in complex analysis.

[2]Whittaker, E.T. A Course of modern analysis / E. T. Whittaker. L. : Cambridge University Press, 1902. P. 63-70.

Holomorphic functions are also sometimes referred to as regular functions or as conformal maps. A holomorphic function whose domain is the whole complex plane is called an entire function[3]. The phrase ”holomorphic at a point z_0” means not just differentiable at z_0, but differentiable everywhere within some neighborhood of z_0 in the complex plane. Given a complex-valued function f of a single complex variable, the derivative of f at a point z_0 in its domain is defined by

$$f'(z) = (\frac{f(z) - f(z_0)}{z - z_0}), \text{where } \lim z \to z_0. \tag{5}$$

Liouville's theorem, named after Joseph Liouville, states that ”every bounded entire function must be constant. A function f defined on some set X with real or complex values is called bounded, if the set of its values is bounded. In other words, there exists a real number M such that

$$|f(z)| \leq M, \text{for all } z \text{ in } C \text{ is constant.}$$

Intuitively, a meromorphic function is a ratio of two well-behaved(holomorphic) functions. Such a function will still be well-behaved, except possibly at the points where the denominator of the fraction is zero. If the denominator has a zero at z and the numerator does not, then the value of the function will be infinite; if both parts have a zero at z, then one must compare the multiplicities of these zeros.

A doubly periodic function with periods $2\omega_1$ and $2\omega_2$ such that

$$f(z+2\omega_1) = f(z+2\omega_2) = f(z).$$

which is analytic and has no singularities except for poles in the finite part of the complex plane. The half-period ratio $\tau = \frac{\omega_1}{\omega_2}$ must not be purely real, because if it is, the function reduces to a singly periodic function if τ is rational, and a constant if τ is irrational (Jacobi 1829), ω_1 and ω_1 are labeled such that:

$$I_{[\tau]} \equiv I[\frac{\omega_1}{\omega_2}] > 0, \text{where } I_{[\tau]} \text{ is imaginary part.}$$

A ”cell” of an elliptic function is defined as a parallelogram region in the complex plane in which the function is not multi-valued. Properties obeyed by elliptic functions include:

Theorem 1.1. The number of poles of an elliptic function $f(z)$ in any cell is finite. (Copson, 1935). If there were an infinite number, then the set of these poles would have a limit point. But the limit point of poles is an essential singularity, and so by definition the function would not be an elliptic function.

[3]Bottazzini, U. Hidden harmony geometric fantasy: The rise of complex function theory / U. Bottazzini, J. Gray. N. Y. : Springer-Verlag, 2013. P. 15-27.

Theorem 1.2. The number of zeroes of an elliptic function $f(z)$ in any cell is finite. (Whittaker and Watson, 1927). If there were an infinite number, then it would follow that the reciprocal of the function would have an infinite number of poles. Therefore, it would have an essential singularity, and this would also be an essential singularity of the original function. Again, this would mean that the function was not an elliptic function.

Theorem 1.3. The sum of all residues of an elliptic function in any period parallelogram is zero.

Let us choose the parallelogram P spanned by $\{\omega_1, \omega_2\}$. Then

$$C = U_{\omega \in L}(P + \omega).$$

Let C be the closed curve $C = \partial P$ with positive orientation. The residue of an elliptic function f in P is

$$\int_c f(z)dz = \int_1 f(z)dz + \int_2 f(z)dz + \int_3 f(z)dz + \int_4 f(z)dz.$$

Here C_1 is the path connecting 0 and ω_1 , C_2 is the path connecting ω_1 and ω_1 + ω_2 and C_3 is the point connecting ω_1 + ω_2 and ω_2 and C_4 is the path connecting ω_2 and 0. We also assume that f has no poles on C: (If f has poles on C, we can use another parallelogram such that the poles of f do not lie on its boundary.) Using the periodicity of f, we obtain

$$\int_1 f(z)dz + \int_2 f(z)dz = \int_3 f(z)dz + \int_4 f(z)dz = 0.$$

This shows that

$$\sum_{p \in P} res_p(f) = \int_c f(z)dz = 0.$$

Corollary 1.1. The number of zeros of a non constant elliptic function in a period parallelogram P is equal to the number of poles in P. The zeros and poles are counted according to their multiplicities[4]. To make it clear let's say f be a non constant elliptic function. Then $\frac{f'(z)}{f(z)}$ is also an elliptic function. By argument principle, the number of zeros minus the number of poles of f in P equals to the sum of residues of f in P, in other words,

$$\frac{1}{2\pi i}\int_{\partial p} \frac{f'(z)}{f(z)} dz = \text{number of zeros of } f(z) \text{ in } P \text{ - number of poles of } f(z) \text{ in } P.$$

Theorem 1.4. An elliptic function without poles is a constant. An elliptic function f with

[4] Koblitz, N. Introduction to elliptic curves and modular forms / N. Koblitz. N. Y. : Springer-Verlag, 1984. P. 14-21.

poles is a bounded entire function. By Liouville's theorem, f must be a constant.

Proposition 1.1. The number of $\frac{zeros}{poles}$ of an elliptic function in any cell is finite. Assume $\{z_n\}g$ is a sequence of poles of an elliptic function f with $z_i \neq z_j$. Since a cell is bounded, Bolzano-Weierstrass theorem implies that $\{z_n\}g$ has a limit point. The limit point would be an essential singularity of f which leads to a contradiction to the assumption that f is a meromorphic function. Notice that the zeros of f are poles of $\frac{1}{f}$: Since $\frac{1}{f}$ is again elliptic, $\frac{1}{f}$ can have only finite many poles in a cell, i.e. f has only finitely many zeros in a cell.

Theorem 1.5. An elliptic function $f(z)$ of order m has m zeroes in each cell. (Copson, 1935). If $f(z)$ is of order m and has n zeroes in a cell, counted with multiplicity, then $(n-m)$ is equal to the sum of residues of $\frac{f'(z)}{f(z)}$ at its poles in the cell. But $f'(z)$ is an elliptic function with the same periods as $f(z)$, so it follows that $\frac{f'(z)}{f(z)}$ is similarly an elliptic function. Therefore, $n-m=0$ by Theorem 1.3.

Theorem 1.6. The sum of the zeros of a non constant elliptic function in a period-parallelogram differs from the sum of its poles by a period. Let's say for example P, C, and C_i ; for $i=1,2,3$ be as that in Theorem 1.2. Suppose $a_1,.....,a_n$ and $b_1,....,b_n$ are zeros and poles of an elliptic function f respectively[5]: Then

$$\sum_{i=1}^{n} a_i - \sum_{j=1}^{n} bj = \frac{1}{2\pi i}\int_c \frac{f'(z)}{f(z)}dz.$$

Using the periodicities of f,

$$\int_{c_1} z\frac{f'(z)}{f(z)}dz + \int_{c_3} z\frac{f'(z)}{f(z)}dz = \int_{c_1}(z-(z+\omega_2)\frac{f'(z)}{f(z)}dz = \omega_2\int_{c_1} z\frac{f'(z)}{f(z)}dz.$$

$$\int_{c_2} z\frac{f'(z)}{f(z)}dz + \int_{c_4} z\frac{f'(z)}{f(z)}dz = \int_{c_4}(z-(z+\omega_1)\frac{f'(z)}{f(z)}dz = \omega_2\int_{c_4} z\frac{f'(z)}{f(z)}dz.$$

This implies that

$$\sum_{i=1}^{n} a_i - \sum_{j=1}^{n} bj = \frac{1}{2\pi i}(\omega_2\int_0^{\omega_2}\frac{f'(z)}{f(z)}dz - \omega_1\int_0^{\omega_1}\frac{f'(z)}{f(z)}dz).$$

Then we can sees that both

$$\frac{1}{2\pi i}\int_0^{\omega_1}\frac{f'(z)}{f(z)}dz.$$

and

$$\frac{1}{2\pi i}\int_0^{\omega_2}\frac{f'(z)}{f(z)}dz,$$

[5] Koblitz, N. Introduction to elliptic curves and modular forms / N. Koblitz. N. Y. : Springer-Verlag, 1984. P. 14-21.

are integers.

In fact, choosing a branch of log; we have

$$\int_0^{\omega_j} \frac{f'(z)}{f(z)} dz = log(\frac{f'(z)}{f(z)}).$$

Since $f(\omega_j) = f(0), we find \int_0^{\omega_j} \frac{f'(z)}{f(z)} dz = log(1) = 2n_j\pi i$, for some integers n_j. This implies that

$$\sum_{i=1}^{n} a_i - \sum_{j=1}^{n} bj = m\omega_1 + n\omega_2, \text{for } n, m \in Z.$$

Theorem 1.7. If $f(z)$ and $g(z)$ are elliptic functions with poles at the same points, and with the same principal parts at these points, then $f(z) = g(z) + A$, for some constant A . (Jones and Singerman, 1987). The function $f(z) - g(z)$ is an elliptic function of order zero, as it has no poles.

Theorem 1.8. If $f(z)$ and $g(z)$ are elliptic functions with zeroes and poles of the same order at the same points, then
$f(z) = Ag(z)$, for some constant A. (Jones and Singerman, 1987). By a similar argument to the previous proof, we have that the function $\frac{f(z)}{g(z)}$ is also an elliptic function of order zero.

Definition 1.1. The order of an elliptic function is the number of poles counting orders, modulo its lattice. An elliptic function has order 0 if and only if it is a constant function. There are no elliptic functions of order 1, since that would entail having a residue that is both zero and non-zero because it's at a pole of order one. For order 2, there are two possibilities one pole of order 1 with residue 0 or two poles of order 1 with residues that are additive inverses of one another. By Mittag-Leffler's theorem, functions satisfying both cases exist[6]. For the first case, it is tempting to guess

$$\sum_{\omega \in L} \frac{1}{(z-\omega)^2}.$$

but this series is neither absolutely nor uniformly convergent on compact subsets of $\frac{C}{L}$, which means it is not necessarily periodic nor meromorphic. This guess can be salvaged by introducing some correcting terms, and in fact this is what Weierstrass himself did. The correcting terms are analogous to those found in the partial fraction decompositions of functions such as the secant.

Definition 1.2. The Weierstras function with associated lattice L is given by the following

[6]Koblitz, N. Introduction to elliptic curves and modular forms / N. Koblitz. N. Y. : Springer-Verlag, 1984. P. 14-21.

equation for $z \in L$:

$$\wp(z) = \frac{1}{z^2} + \sum_{\omega \in L(0)} [\frac{1}{(z-\omega)^2} + \frac{1}{\omega^2}].$$

Theorem 1.9.. The Weierstrass function ($\wp$) is an even elliptic function. It can be shown that the series is absolutely convergent and that the series is uniformly convergent on compact subsets of $\frac{C}{L}$. One method uses estimation techniques from analysis and comparison with an integral. Absolute convergence shows that the function is even, since replacing z with $-z$ simply rearranges the terms. Uniform continuity shows that the function is meromorphic, since the individual terms are meromorphic.
For periodicity on L, the derivative is as follows, which is clearly periodic.

$$\wp'(z) = -2 \sum_{\omega \in L(0)} \frac{1}{(z-\omega)^3}.$$

We know that $\wp(z)$ and $(z+\omega_1)$ differ by a constant because they have the same derivative, and this constant can be seen to be 0 by taking $z = \frac{1}{2}(\omega_1)$. The proof can be completed by repeating this with ω_2.

The man who first studied the lemniscatic integral from a purely functional point of view was Legendre.In his paper he showed for the first time how any integral of the form[7]: $\int \frac{Q(z)dz}{R(z)}$, where $Q(z)$ is a rational function in z and $R(z)$ is the square root of a quartic (with real coefficients), can be reduced to one of the following form:

$$\int \frac{Q(z)dz}{\sqrt{(1-z^2)(1-k^2z^2)}}. \tag{6}$$

In the above equation, if we substitute $z = sin(\phi)$ it farther reduces to the following form:

$$\int \frac{Q(z)d\phi}{\sqrt{(1-k^2sin^2(\phi))}}. \tag{7}$$

The variable ϕ Legendre called the amplitude of the elliptic integral, the parameter k the modulus, and quantity b defined as $\sqrt{1-k^2}$ the complementary modulus. As Legendre did, denoting: $\Delta = \sqrt{(1-k^2sin^2\phi)}$, Legendre showed the three distinct kinds of elliptic integrals.

$$\int_0^x \frac{d\phi}{\Delta}. \tag{8}$$

[7]Hancock, H. Lectures on the theory of elliptic functions / H. Hancock. N.Y.: Wiley, 1910. P. 1-6.

$$\int_0^x \Delta d\phi. \tag{9}$$

$$\int_0^x \frac{d\phi}{(1+n^2sin^2\phi)\Delta}. \tag{10}$$

Where n may be real or complex, and according to Legendre's terminology he called them first kind, second kind and third kind respectively. All the integrals were regarded as functions of their upper end point x. He wrote F for a typical integral of the first kind and E or G for one of the second kind. The complete integrals $\int_0^{\frac{\pi}{2}} \frac{d\phi}{\Delta}$ and $\int_0^{\frac{\pi}{2}} \Delta d\phi$ he denoted F' and E', respectively, or $F'(k)$ and $E'(k)$ when he wanted to think of them as functions of the modulus k. He said it was indispensable that the modulus k and the amplitude ϕ were real, and that $k < 1$.

Legendre's elliptic integrals of the first kind:

- The incomplete integrals of the first kind are of the form:

$$F(\varphi,k) = \int_0^{\varphi} \frac{dt}{\sqrt{(1-k^2sin^2t)}} = \int_0^{sin(\varphi)} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{11}$$

- The complete integrals of the first kind are of the form:

$$F(\frac{\pi}{2},k) = K(k) = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{(1-k^2sin^2t)}} = \int_0^1 \frac{dt}{\sqrt{(1-x^2)(1-k^2x^2)}} = F \tag{12}$$

$$F(\frac{\pi}{2},k') = \int_0^{\frac{\pi}{2}} \frac{dt}{\sqrt{(1-k'^2sin^2t)}} = F(k') = F \tag{13}$$

Legendre's elliptic integrals of the second kind:

- The incomplete integrals of the second kind are of the form:

$$E(\varphi,k) = \int_0^{\varphi} \sqrt{(1-k^2sin^2t)}dt = \int_0^{sin(\varphi)} \frac{\sqrt{1-k^2x^2}}{\sqrt{(1-x^2)}}dx. \tag{14}$$

- The complete integrals of the second kind are of the form:

$$E(k) = \int_0^{\frac{\pi}{2}} \sqrt{(1-k^2sin^2t)}dt = \int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{(1-x^2)}}dx. \tag{15}$$

or

$$E=\int_0^1 \frac{\sqrt{1-k^2x^2}}{\sqrt{1-x^2}}dx. \tag{16}$$

$$E'=\int_0^1 \frac{\sqrt{1-k'^2x^2}}{\sqrt{1-x^2}}dx. \tag{17}$$

If we put $d\phi = dn(u)du$, $\Delta\phi = dn(u)$, we get:
$E(k,\phi)=E(u)=\int_0^u (dn^2u)du=\int_0^u (1-k^2sn^2u)du$

Legendre's elliptic integrals of the third kind:

- The incomplete integrals of the third kind are of the form:

$$\pi(n;\varphi,k)=\int_0^{\varphi} \frac{dt}{(1+n.sin^2t)\sqrt{(1-k^2sin^2t)}}. \tag{18}$$

$$\pi(n;\varphi,k)=\int_0^{sin(\varphi)} \frac{dt}{(1+nx^2)\sqrt{(1-x^2)(1-k^2x^2)}}. \tag{19}$$

- The complete integrals of the third kind are of the form:

$$\pi(n,k)=\int_0^{\frac{\pi}{2}} \frac{dx}{(1-n.sin^2x)\sqrt{(1-k^2sin^2x)}}. \tag{20}$$

$$\pi'(n,k)=\int_0^{\frac{\pi}{2}} \frac{dx}{(1+n.sin^2x)\sqrt{(1-k^2sin^2x)}}. \tag{21}$$

The Jacobian sn, cn and dn functions. Suppose we have the two integrals (a):$u=\int_0^x \frac{dt}{\sqrt{1-t^2}}$, $(b):$ $\frac{1}{2}\pi=\int_0^1 \frac{dt}{\sqrt{1-t^2}}$, where $-1<x<1$ is real. If we take the square root to be positive for u between 0 and π, then this defines u as an odd function of x. By inversion of the integral, we have defined z as an odd function of u. If denote this function by $sin(u)$, then (a) reduces to the form $u=sin^{-1}x$. We can define a second function of $cos(u)$ by $cos(u)=\sqrt{1-sin^2(u)}$. By taking the square root positive for u between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$, ?we have u as an even function of x. It follows that we have the identity $sin^2(u)+cos^2(u)=1$. We can also note that $sin(0)=0$ and $cos(0)=1$.
Suppose now we consider the derivative of (a) with respect to x, which is clearly $\frac{du}{dx}=\frac{1}{\sqrt{1-x^2}}$. It follows that $\frac{d}{du}\{sin(u)\}=\sqrt{1-sin^2(u)}=cos(u)$, as $x=sin(u)$. Moreover, by differentiation (b), we obtain $\frac{d}{du}\{sin(u)\}=-sin(u)$. Now, let us see Jacobi elliptic function

as functions of the complex variable u, the Jacobi elliptic functions $sn(u)$,$cn(u)$ and $dn(u)$ are doubly periodic single valued functions of u.

i. The inverse function of the Legendre elliptic function F is $\phi = F^{-1}(u)$. and the Jacobi elliptic function $sin(u) = sin(\phi)$ or $sn(u|n)$ is single valued for all complex parameters with

$$u = \int_0^{sn(u)} \frac{du}{\sqrt{(1-t^2)(1-kt^2)(u)}}, \tag{22}$$

and it is analogous to the function $sin(u)$ by the relation

$$u = \int_0^{sin(u)} \frac{dt}{\sqrt{(1-t^2)}}, \tag{23}$$

$sn(u)$ is an odd single valued doubly-periodic function of u, with two poles in each period-parallelogram, the distance between the poles being half of one of the periods. The two periods will be connected by a relation, as they depend only on the single constant k, and the constant parameter k is called modulus.

ii. The Jacobi elliptic function cn is defined by $cn(u) = cn(u|k) = cos(\varphi).cn(u) = \sqrt{1-sn^2(u)}$ and $cn(u)$ is an even single valued function of u.

iii. The Jacobi elliptic function dn is defined by $dn(u) = dn(u|k) = \sqrt{1-k^2sn^2(u)}$ and $dn(u)$ is also a single valued function of u.This is because both $cn(u)$ and $dn(u)$ have definite values at a point $u = 0$, it follows that the functions $cn(u)$ and $dn(u)$ are single valued functions of u. They obviously satisfy the relations:

$$sn^2(u) + cn^2(u) = 1 \tag{24}$$

and

$$k^2 sn^2(u) + dn^2(u) = 1. \tag{25}$$

The functions $sn(u), cn(u), dn(u)$ are often called the Jacobian elliptic functions. Now, let us see the expression of $cn(u)$ and $dn(u)$ by means of integrals,so by differentiating the equation $cn^2(u) = 1 - sn^2(u)$, then we get:

$cn(u)\frac{d}{du}cn(u) = -sn(u)cn(u)dn(u)$,then we get: $\frac{d}{du}cn(u) = -sn(u)dn(u)$ and this is equal to the equation $-\sqrt{(1-cn^2(u))(k'^2+k^2cn^2(u))}$, where $k'^2 = 1-k^2$.

Thus let us put $cn(u) = r$, then $du = \frac{dr}{\sqrt{(1-r^2)(k'^2+k^2r^2)}}$, therefore by putting $u = 0$ we have $cn(u) = cn(0) = 1$, and by integrating both sides we get:

$$u = \int_{cn(u)}^{1} \frac{dr}{\sqrt{(1-r^2)(k'^2 + k^2 r^2)}}. \tag{26}$$

Similarly $\frac{d}{du} dn(u) = -k^2 sn(u) cn(u)$, then we get:

$$u = \int_{dn(u)}^{1} \frac{dr}{\sqrt{(1-r^2)(r^2 - k'^2)}}. \tag{27}$$

The Jacobi elliptic functions are arise from the inversion of the elliptic integral of the first kind[8]. Let us now denote the three basic functions as: $sn(u,k), cn(u,k)$ and $dn(u,k)$, where k is the elliptic modulus. From the elliptic integral of the first kind we have $u = F(\phi, k) = \int_0^{\phi} \frac{dt}{\sqrt{1-k^2 sin^2(t)}}$,where $0 < k^2 < 1$, $k = mod(u)$ is the elliptic modulus, and $\phi = am(u,k) = am(u)$ is the Jacobi amplitude, given by: $\phi = F^{-1}(u,k) = am(u,k)$. Then we get:

$$sin(\phi) = sin(am(u,k)) = sn(u,k) cos(\phi) = cos(am(u,k)) = cn(u,k) \sqrt{(1-k^2 sin^2(\phi))} = \sqrt{(1-k^2 si}$$

and these functions are doubly-periodic functions satisfying:
$sn(u,\theta) = sin(u), cn(u,\theta) = cos(u)$ and $dn(u,\theta) = 1$.
Periodicity of the Jacobian functions. Now let us see the constant K' and the periodicity of $K, K+iK'$ and iK' with the help of the addition-theorem[9]:

- Periodicity of the elliptic functions with respect to K, the constant K is intimately connected with the periodicity of the elliptic functions $sn(u), cn(u), dn(u)$. Therefore we can see that, $4K$ is a period for the functions $sn(u)$ and $cn(u)$, and $2K$ is a period for the function $dn(u)$.

- The constant K', let us denote the following integral by K', but the sign of i must be understood in the light of the relation which was used in the transformation. This lead us to the results
$s = sn(K+iK') = \frac{1}{k}$, and $cn(K+iK') = \frac{ik}{k'}$.

- Periodicity of the elliptic functions with respect to $K+iK'$, we have seen that, K' has an important connection with the second period of the functions $sn(u), cn(u), dn(u)$.

[8]Bottazzini, U. Hidden harmony geometric fantasy: The rise of complex function theory / U. Bottazzini, J. Gray. N. Y. : Springer-Verlag, 2013. P. 15-27.

[9]Cayley, A. An elementary treatise on elliptic functions / A. Cayley. London : Bell and Sons, 1895. P. 1-20.

Therefore, the function $cn(u)$ has a period at $2K+2iK'$ and the functions $sn(u)$ and $dn(u)$ have a period at $4K+4iK'$.

- Periodicity of the elliptic functions with respect to iK', Therefore we get that, the function $sn(u)$ has a period at $2iK'$ and the functions $cn(u)$ and $dn(u)$ have a period at $4iK'$.

- The behavior of the functions $sn(u), cn(u), dn(u)$ at the point $u = iK'$, for the points neighborhood the point $u = 0$,the function $sn(u)$ can be expanded by Taylor's theorem. Therefore this shows that at the point $u = iK'$ the functions $sn(u), cn(u)$ and $dn(u)$ a simple poles, with the residues
$\frac{1}{k}, \frac{-i}{k}$ and $-i$ respectively.

Conclusion. Legendre's work, was a simplification of the general elliptical integral, and the subsequent computation of the values of elliptical integrals as functions of the coefficients and their upper end points. The analogy between the trigonometric and elliptic integrals is helpful to compute tables of, say, the sine function, one would make repeated use of the addition formula.

In the other hand, as a result of Jacobian functions we get the following points periodicity results:

- $sn(u)$ is a single valued doubly-periodic function of u, its periods being $4K$ and $2iK'$. Its singularities are at all points congruent with $u = iK'$ and $u = 2K + iK'$; they are simple poles, with the residues $\frac{1}{k}$ and $-\frac{1}{k}$ respectively; and the function is zero at all points congruent with $u = 0$ and $u = 2K$. When k^2 is real and positive and less than unity, it is easily seen that K and K' are real, and $sn(u)$ is real for real values of u and purely imaginary for purely imaginary values of u.

- $cn(u)$ is a single valued doubly-periodic function of u its periods being $4K$ and $2K + 2iK'$. It's singularities are at all points congruent with $u = iK'$ and $u = 2K + iK'$, they are simple poles, with the residues $\frac{i}{k}$ and $-\frac{i}{k}$ respectively; and the function is zero at all points congruent with $u = K$ and $u = 3K$.

- $dn(u)$ is a single valued doubly-periodic function of u, its periods being $2K$ and $4iK'$. Its singularities are at all points congruent with $u = iK'$ and $u = 3iK'$; they are simple poles, with the residues $-i$ and $+i$ respectively; and the function is zero at all points congruent with $u = K + iK'$ and $u = K + 3iK'$.